

ESSENTIAL DIMENSION LOWERING MAPPINGS HAVING DENSE DEFICIENCY SET

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ABSTRACT. Two classes of surjective maps $f: S^m \rightarrow S^n$ that are one-to-one over the image of a dense set are constructed. We show that for $m, n \geq 3$ there is a monotone surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set; and for $3 \leq n \leq m \leq 2n - 3$, each element of $\pi_m(S^n)$ can be represented as a monotone surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set.

1. Introduction. The present paper should be considered as a continuation of the study of surjective maps between spheres that are “one-to-one over the image of a dense set”. (A surjection $f: X \rightarrow Y$ is *one-to-one over the image of a dense set* if there exists a dense set $D \subseteq X$ such that for each $y \in f(D)$, $\#f^{-1}(y) = 1$; $\# =$ cardinality.)

First inconceivable examples of such maps were constructed by J. J. Walsh [Wa 5]. Specifically, for any pair $n \geq 3$, $d \geq 2$ of integers, Walsh has built a monotone, surjective map $f: S^n \rightarrow S^n$ of degree d that is one-to-one over the image of a dense set.

More recently, in [Be-Wa], it has been established that for any $m, n \geq 2$ there is a surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set. By construction this map is not monotone and factors through a 1-dimensional compactum, and hence it is null-homotopic; even more, it has no stable values. (A point $y \in Y$ is a *stable value* of a map $f: X \rightarrow Y$ between metric spaces if there exists an open cover $\mathcal{U}$ of Y so that for every $\mathcal{U}$ -approximation f' to f , y is in the image of f' .)

In this paper we show:

(a) For $m, n \geq 3$ there is a *monotone* surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set.

(b) For $3 \leq n \leq m \leq 2n - 3$ each element of $\pi_m(S^n)$ can be represented as a *monotone* surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set. In particular, if $3 \leq n \leq m \leq 2n - 3$ and $\pi_m(S^n) \neq 0$ (e.g., $\pi_{n+1}(S^n) = \mathbb{Z}_2$), there is a monotone, *essential* map $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set (and hence, all values of f are stable).

The techniques used in the paper stem from D. Wilson [Wi 1, Wi 2] and J. J. Walsh [Wa 1–Wa 5]. Mappings are constructed by making use of “defining sequences”. Although the necessary definitions are given, and in that respect the

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paper is self-contained, familiarity with [Wa 5] is desirable. We follow the notation developed in that paper.

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2. Preliminaries. For any family P of subsets of a set X , and for any $A \subseteq X$, we define

$$\text{St}(A, P) = \bigcup \{p \in P : p \cap A \neq \emptyset\}.$$

Following [Wa 5], by a (*stratified*) *partition* on a closed PL n -manifold N we mean a collection $P = \{p_1, \dots, p_k\}$ of closed subsets of N that cover N with the following properties.

(P1) Each $p \in P$ is a PL n -submanifold (with boundary) of N .

(P2) If $p_{i(1)}, \dots, p_{i(t)}$ are mutually distinct elements of P , then $p_{i(1)} \cap \dots \cap p_{i(t)}$ is either empty or an $(n - t + 1)$ -dimensional PL submanifold of the boundary of $p_{i(1)} \cap \dots \cap p_{i(t-1)}$.

Observe that $p_{i(1)} \cap \dots \cap p_{i(t)} \neq \emptyset$ has empty boundary if and only if $p \cap p_{i(1)} \cap \dots \cap p_{i(t)} = \emptyset$ for all $p \in P - \{p_{i(1)}, \dots, p_{i(t)}\}$.

If L is any triangulation of N , by J_i^N denote the standard handlebody decomposition of N associated with the i th barycentric subdivision $\beta^i L$ of L :

$$J_i^N = \{\text{St}(v, \beta^{i+1} L) : v \text{ is a vertex of } \beta^i L\}.$$

It is easy to see that J_i^N satisfies (P1) and (P2). For $i \geq 1$ and $j = \text{St}(v, \beta^{i+1} L) \in J_i^N$, define the *index* of j , $\text{Ind}(j)$, to be equal to k if v is the barycenter of a k -simplex in $\beta^i L$.

Let M^m, N^n be PL manifolds, and P, Q partitions on M, N respectively. We say that a function $T: P \rightarrow Q$ is *admissible*, provided:

(A1) T is a bijection;

(A2) for all $p_{i(1)}, \dots, p_{i(t)} \in P$,

$$p_{i(1)} \cap \dots \cap p_{i(t)} \neq \emptyset \Rightarrow T(p_{i(1)}) \cap \dots \cap T(p_{i(t)}) \neq \emptyset;$$

(A3) for all $p, p' \in P$,

$$p \cap p' \neq \emptyset \Rightarrow T(p) \cap T(p') \neq \emptyset.$$

Let L be any triangulation of N . If $T: P \rightarrow J$ is a triple satisfying (A2) and (A3), where $J = J_i^N$ is the handlebody decomposition of N associated with $\beta^i L$, by an *induced map* we mean any map $h: M \rightarrow N$ with $h(p) \subseteq T(p)$ for all $p \in P$. We can define h by the “backward induction” on t , requiring that $h(p_{i(1)} \cap \dots \cap p_{i(t)}) \subseteq T(p_{i(1)}) \cap \dots \cap T(p_{i(t)})$. Since each nonempty intersection of elements of J is an absolute retract, the inductive step “goes through”. The same fact establishes that any two induced maps are homotopic (see [Wa 5]), which enables us to talk about the induced map.

A sequence of triples $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ is a *defining sequence* provided, for all $i \geq 0$:

(DS1) $J_i = J_i^N$ is the handlebody decomposition of N associated with $\beta^i L$;

(DS2) P_i is a partition on M ;

(DS3) T_i is an admissible function;

(DS4) for all $p \in P_i$, $p' \in P_{i+1}$,

$$p \cap p' \neq \emptyset \Leftrightarrow \text{Int}(p \cap p') \neq \emptyset \Leftrightarrow T_i(p) \cap T_{i+1}(p') \neq \emptyset.$$

The reader should find establishing the following result a useful exercise.

PROPOSITION 2.1 (SEE [Wa 5]). *Let $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ be a defining sequence.*

(i) *Setting $h^{-1}(y) = \bigcap_{i=0}^\infty \text{St}(p_i, P_i)$ for any choice of $p_i \in P_i$ with $T_i(p_i) \ni y$ defines a surjective map $h: M \rightarrow N$. Moreover, $\text{Int St}(p_i, P_i) \supseteq \text{St}(p_{i+1}, P_{i+1})$ ($i = 0, 1, 2, \dots$).*

(ii) *If $h_i: M \rightarrow N$ is the map induced by $T_i: P_i \rightarrow J_i$, then $h = \lim_{i \rightarrow \infty} h_i$ and $h_0 \simeq h_1 \simeq h_2 \simeq \dots \simeq h$.*

(iii) *If each $p \in P_i$, $i \geq 0$, is connected, then h is a monotone map.*

(iv) *If for each $i \geq 1$ and each $j \in J_i$ with $\text{Ind}(j) = n$ there exists a PL m -cell $B \subseteq M$ with*

$$\text{St}(T_i^{-1}(j), P_{i+1}) \subseteq \text{Int } B \subseteq \text{St}(T_i^{-1}(j), P_i),$$

then the points $y \in N$ for which $h^{-1}(y) \subseteq M$ is a cellular set form a dense subset of N .

To construct interesting maps between manifolds using 2.1, we have to produce defining sequences. The major step consists of generating a triple $T_{i+1}: P_{i+1} \rightarrow J_{i+1}$ from a triple $T_i: P_i \rightarrow J_i$ previously constructed. To make the notation easier, the triple $T_i: P_i \rightarrow J_i$ will be denoted by $T: P \rightarrow J$, and the triple $T_{i+1}: P_{i+1} \rightarrow J_{i+1}$ by $\hat{T}: \hat{P} \rightarrow \hat{J}$. Coherently, we will rename the subdivision $\beta^i L$ and again call it L . Hence

$$J = \{\text{St}(v, \beta L): v \text{ is a vertex of } L\},$$

$$\hat{J} = \{\text{St}(v, \beta^2 L): v \text{ is a vertex of } \beta L\}.$$

The construction of $\hat{P}$ is in two stages. We define an intermediate triple $\hat{T}: \hat{P} \rightarrow \hat{J}$. *Warning.* $\hat{P}$ will be a partition of M , and $\hat{T}$ will satisfy (A2) and (A3), but not necessarily (A1).

The elements of $\hat{P}$ will be indexed by the set S of all collections $\{p_{i(1)}, \dots, p_{i(t)}\} \subseteq P$ that have nonempty intersections. (These intersections, in Walsh's terminology, are called the *strata* of P .)

The collection $\hat{P} = \{p_s, s \in S\}$ will satisfy the following properties.

(H1) $\hat{P}$ is a partition on M .

(H2) $p_{s(1)}, \dots, p_{s(t)} \in \hat{P}$ have nonempty intersection if and only if $\{s(1), \dots, s(t)\} \subseteq S$ is well-ordered with respect to inclusion.

(H3) For any $p \in P$ and $p_s \in \hat{P}$,

$$p \cap p_s \neq \emptyset \Leftrightarrow \text{Int}(p \cap p_s) \neq \emptyset \Leftrightarrow p \in s.$$

(H4) For any $p \in P$, p and $p_{\{p\}}$ are homeomorphic.

Still following [Wa 5], we construct the elements $p_s \in \hat{P}$ as follows (see Figure 1). Let K be a triangulation of M so that each stratum $\bigcap s$, $s \in S$, is a full subcomplex of K . Define the *core* of $s \in S$ by

$$c(s) = \bigcup \left\{ \tau: \tau \text{ is a simplex of } \beta K \text{ contained in } \bigcap s - \partial \left(\bigcap s \right) \right\}.$$

Finally, set

$$p_s = \bigcup \{ \text{St}(v, \beta^2 K): v \in c(s) \text{ is a vertex of } \beta K \}.$$

Observe that, by choosing a sufficiently small triangulation K :

(H5) Given neighborhoods $U(p)$ of $p \in P$, we can arrange that $\text{St}(p, \hat{P}) \subseteq U(p)$ for all $p \in P$.

We can also define the function $\hat{T}: \hat{P} \rightarrow \tilde{J}$ by $\hat{T}(p_s) = \text{St}(v, \beta^2 L) \in \tilde{J}$, where v is determined as follows. If $s = \{p_{i(1)}, \dots, p_{i(t)}\}$ and $T(p_{i(r)}) = \text{St}(v_r, \beta L)$, then v is the barycenter of the simplex whose vertices are $v_1, \dots, v_t$. Property (H2) implies that $\hat{T}$ satisfies (A2) and (A3). It is evident that $\hat{T}$ is a one-to-one function (but not necessarily a surjection).

We will “repair” the triple $\hat{T}: \hat{P} \rightarrow \tilde{J}$ to get the triple $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$, but the reparation will depend on the desired properties of the function $h: M \rightarrow N$ determine by the defining sequence. The “reparation process”, as well as the construction of the triple $T_0: P_0 \rightarrow J_0$, is explained in detail in forthcoming sections.

REMARK. If the partition P is the standard handlebody decomposition corresponding to a triangulation K of the manifold M , then the partition $\hat{P}$ constructed above is (up to an ambient isotopy) the standard handlebody decomposition of M corresponding to the barycentric subdivision K' of K . This fact will be implicitly used in the sequel.

3. Essential maps. The purpose of this section is to establish the following

PROPOSITION 3.1. *Let $f: S^m \rightarrow S^n$ be any map, and let $3 \leq n \leq m \leq 2n - 3$. Then there exists a surjective monotone map $h: S^m \rightarrow S^n$ homotopic to f such that the set $\{y \in S^n: h^{-1}(y) \text{ is cellular in } S^m\}$ is dense in S^n .*

A routine consequence of 3.1 is the result announced in the Introduction.

THEOREM 3.2. *For any map $f: S^m \rightarrow S^n$, $3 \leq n \leq m \leq 2n - 3$, there exists a surjective monotone map $g: S^m \rightarrow S^n$ homotopic to f that is one-to-one over the image of a dense set.*

PROOF. Let $h: S^m \rightarrow S^n$ be a map whose existence is promised by 3.1. We “carefully shrink countably many cellular fibers of h ” in order to obtain the sought-after map $g: S^m \rightarrow S^n$. The shrinking process can be described as follows.

Let $U_1, U_2, \dots$ be a countable basis of open sets of S^m . Choose a fiber F of h with $F \cap U_1 \neq \emptyset$, and pick a cellular fiber C of h in a “small” *connected* neighborhood V of F (here we use the fact that h is a monotone map). Let $\lambda: S^m \rightarrow S^m$ be a surjection whose only nondegenerate point-preimage is C . We can arrange that $\lambda(C) \in U_1$ and $\lambda = \text{identity}$ off of V . Then $g_1 = h\lambda^{-1}$ is a monotone surjection “close” to h , and one of the fibers of g_1 is a point in U_1 . In a similar fashion we produce monotone maps $g_2, g_3, \dots$ such that g_{i+1} is “close” to g_i , it agrees with g_i off of a “small” neighborhood of a fiber of g_i , and g_{i+1} has degenerate point-preimages in each of the sets $U_1, \dots, U_{i+1}$.

Exercising sufficient control on all choices made, and carefully interpreting the quoted words in the preceding paragraph, we can arrange that the sequence $g_1, g_2, \dots$ converges to a monotone map $g: S^m \rightarrow S^n$ homotopic to h that is one-to-one over the image of a dense set.

Before giving a proof of Proposition 3.1, we state and prove an interesting corollary of Theorem 3.2.

In what follows, E^r denotes Euclidean r -dimensional space. Observe that the homogeneity properties of E^r establish that the set of all stable values of a surjection $f: X \rightarrow E^r$ is open in E^r .

COROLLARY 3.3. *Let $m, n \geq 2$ be integers.*

(i) *If $\pi_{m-1}(S^{n-1}) = 0$, then any surjection $f: E^m \rightarrow E^n$ that is one-to-one over the image of a dense set has no stable values.*

(ii) *If $3 \leq n \leq m \leq 2n - 3$ and $\pi_{m-1}(S^{n-1}) \neq 0$, then there exists a proper monotone surjection $f: E^m \rightarrow E^n$ that is one-to-one over the image of a dense set and has all values stable.*

PROOF. (i) In view of the observation made before the statement of Corollary 3.3, it suffices to prove that if $\#f^{-1}(y) = 1$, then y is not a stable value of f . Let B be a “small” ball around $f^{-1}(y)$. The assumption about the homotopy group reveals that $f|_{\partial B}$ is a null-homotopic map in a “small” deleted neighborhood of y . Redefine f in $\text{Int } B$, using the homotopy, to get an approximation f' to f whose image misses y .

(ii) By Freudenthal’s Suspension Theorem (see [Sp, p. 458]), $\pi_{m-1}(S^{n-1}) \cong \pi_m(S^n)$. Application of 3.2 gives an *essential* monotone surjection $f: S^m \rightarrow S^n$ that is one-to-one over the image of a dense set. Pick $y \in S^n$ such that $\#f^{-1}(y) = 1$. Then $f/: S^m - f^{-1}(y) \rightarrow S^n - y$ is a proper monotone surjection that is one-to-one over the image of a dense set. All values of $f/$ are stable since the opposite would violate the fact that f is essential.

PROOF OF 3.1. We construct a defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ with the following additional properties.

(E1) For mutually distinct elements $p_{i(1)}, \dots, p_{i(t)} \in P_i$,

$$p_{i(1)} \cap \dots \cap p_{i(t)} \neq \emptyset \Leftrightarrow t \leq n + 1 \text{ and } T(p_{i(1)} \cap \dots \cap T(p_{i(t)})) \neq \emptyset$$

($i = 0, 1, \dots$).

(E2) Each element $p \in P_i$ is $(m - n)$ -connected ($i = 0, 1, \dots$).

(E3) If $j \in J_i$, $\text{Ind}(j) = n$, then $T_i^{-1}(j)$ is a PL m -ball ($i = 1, 2, \dots$).

As announced in §2, we construct the defining sequence by induction. Suppressing indices, we start with an admissible function $T: P \rightarrow J$ satisfying (E1) and (E2) (produced following the inductive analysis). Let K be a triangulation of S^m such that all strata of P are full subcomplexes of K . Let $\hat{T}: \hat{P} \rightarrow \hat{J}$ be the triple constructed in §2, $\hat{P} = \{p_s, s \in S\}$. Observe that (E1) implies that $\hat{T}$ satisfies (A1)–(A3). Also, (H2) implies that $\hat{T}$ satisfies (E1).

We now “repair” the triple $\hat{T}: \hat{P} \rightarrow \hat{J}$ to get another triple $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$ which satisfies (E2) and (E3). We want to maintain all properties that $\hat{T}: \hat{P} \rightarrow \hat{J}$ already satisfies. For all $s \in S$ choose $p(s) \in s$; if possible, choose $p(s)$ so that $\text{Ind } T(p(s)) = n$. A quick remark: each $s \in S$ contains at most one p with $\text{Ind } T(p) = n$. We interrupt the proof to introduce some notation.

For a compactum X , denote by $C(X) = X \times [0, 1]/(x_1, 1) \sim (x_2, 1)$ the *cone* over X . We identify $X = X \times \{0\} \subseteq C(X)$. Name $\frac{1}{2}C(X) = X \times [0, \frac{1}{2}] \subset C(X)$ the *bottom half of the cone* over X . If A is a subcomplex of K , by $A^{(r)}$ we denote the r -skeleton of A with respect to K . Finally, “ $\approx$ ” means “PL homeomorphic”.

For all $s \in S$ with $\#s > 1$ choose a polyhedron $X_s \subseteq S^m$ containing $c(s)$ with the following properties.

- (a) $X_s \subseteq p(s) \cap (p_s \cup \text{Int } p_{\{p(s)\}})$;
 - (b) $X_s \cap \partial p(s) = c(s)$;
 - (c) $(X_s, X_s \cap p_s, c(s)) \approx ((c(s)) \cup C((c(s))^{(m-n)}), c(s) \cup \frac{1}{2}C((c(s))^{(m-n)}), c(s))$,
- and

- (d) if $s_1 \neq s_2$, then $X_{s_1} \cap X_{s_2} = \emptyset$.

Sets X_s exist, since $\dim C(c(s))^{(m-n)} \leq m - n + 1$, and $2(m - n + 1) < m$. The same inequality, coupled with (c) and the fact that each p_s with $\#s = 1$ is $(m - n)$ -connected (see (H4)), testifies that $p_{\{p(s)\}} \setminus \bigcup \{X_{s'} : s' \in S, \#s' > 1\}$ is still $(m - n)$ -connected for all $s \in S$.

Let K' be a subdivision of K such that all mentioned subsets of S^m are full subcomplexes with respect to K' . Let N_s be the second derived neighborhood of X_s in $p(s)$ with respect to K' . Define

$$\tilde{p}_s = \begin{cases} p_s \cup N_s & \text{if } \#s > 1, \\ p_s \setminus \bigcup \{\text{Int } N_{s'}, s' \in S, \#s' > 1\} & \text{if } \#s = 1. \end{cases}$$

The reader can easily verify that $\tilde{P} = \{\tilde{p}_s, s \in S\}$ is a partition on S^m , that $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$ defined by $\tilde{T}(\tilde{p}_s) = \hat{T}(p_s)$ is an admissible function, and that the triple $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$ satisfies (E1) and (E2). (Note that p_s collapses to $c(s)$; by adding the cone over $c(s)^{(m-n)}$, we “killed” first $(m - n)$ homotopy groups).

If $\#s = n + 1$, then $\tilde{p}_s$ is a regular neighborhood of $X_s \approx C(c(s))$ (since $\dim(c(s)) = m - n$), and hence $\tilde{p}_s$ is an m -ball. This establishes (E3). From (a) and (H3), it easily follows that T and $\tilde{T}$ satisfy (DS4).

Observe that, by our choice of $p(s)$, $s \in S$, we have $\text{St}(p, \tilde{P}) = \text{St}(p, \hat{P})$, for all $p \in P$ with $\text{Ind } T(p) = n$. Thus, taking into account (H5), we get:

(E4) Given neighborhoods $U(p)$ of $p \in P$ with $\text{Ind } T(p) = n$, we can arrange that $\text{St}(p, \tilde{P}) \subseteq U(p)$ for all such p .

If $h: S^m \rightarrow S^n$ is the map associated with a defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ satisfying (E1)–(E3), then, by 2.1(iii), h is a monotone surjection, and using 2.1(iv), together with (E4), we see that we can arrange that the set $\{y \in S^n: h^{-1}(y) \text{ is cellular}\}$ is dense in S^n . Indeed, let B_j be a regular neighborhood of $T_i^{-1}(j)$ contained in $\text{St}(T_i^{-1}(j), P_i)$. By (E3), B_j is an m -ball. By (E4) we can arrange that $\text{St}(T_i^{-1}(j), P_{i+1}) \subseteq \text{Int } B_j$.

To finish the proof of 3.1, in view of 2.1(ii), we need to construct a triple $T_0: P_0 \rightarrow J_0$ satisfying (E1) and (E2) such that the induced map $h_0: S^m \rightarrow S^n$ is homotopic to f . By Freudenthal’s Suspension Theorem [Sp, p. 458], there is a map $f': S^{m-1} \rightarrow S^{n-1}$ whose suspension $\Sigma f': S^m \rightarrow S^n$ is homotopic to f . Without loss of generality, we may assume that f' is a surjective simplicial map, with respect to some triangulations K_0, L_0 of S^{m-1}, S^{n-1} respectively. Then the map $\Sigma f': \Sigma K_0 \rightarrow \Sigma L_0$ is simplicial. To suppress unnecessary symbols, rename it as $f: K \rightarrow L$. Let

$$p_v = \bigcup \{\text{St}(w, \beta K): f(w) = v, w \text{ is a vertex of } K\}$$

and set $P = \{p_v: v \text{ is a vertex of } L\}$. Then P is a partition on S^m , and the function $T: P \rightarrow J_0$ given by $T(p_v) = \text{St}(v, \beta L)$ satisfies (A1)–(A3) and (E1).

We now “repair” the triple $T: P \rightarrow J_0$ to get a new triple $T_0: P_0 \rightarrow J_0$ satisfying, in addition, (E2). The strategy is the same as for obtaining $\tilde{T}$ from $\hat{T}$. Observe

that if σ is a suspension vertex of L , then p_σ is an m -ball. Moreover, p_σ intersects all elements of P except for $p_\tau \in P$, where τ is the other suspension vertex. If $p \in P - \{p_\sigma, p_\tau\}$, then p collapses to $p \cap p_\sigma$. For each $p \in P - \{p_\sigma, p_\tau\}$ choose a polyhedron $A_p \subseteq p \cap p_\sigma$ such that $\dim A_p \leq m - n$ and the pair $(p \cap p_\sigma, A_p)$ is $(m - n)$ -connected. We can arrange that $A_{p(1)} \cap A_{p(2)} = \emptyset$ if $p(1) \neq p(2)$. (We can take A_p to be the $(m - n)$ -skeleton of a shrunk copy of $p \cap p_\sigma$.) Next, embed the cones $C(A_p)$ into p_σ to obtain polyhedra $Y(p)$, $p \in P - \{p_\sigma, p_\tau\}$. We can arrange that $Y(p) \cap \partial p_\sigma = A_p$ for all $p \in P - \{p_\sigma, p_\tau\}$ and, by general positioning, that $Y(p) \cap Y(p') = \emptyset$ for $p \neq p'$ (we are in the range of dimensions where $2(m - n + 1) < m$). Let K' be a subdivision of K such that all mentioned subsets of S^m are full subcomplexes of K' . If N_v is the second derived neighborhood of $Y(p_v)$ in p_σ , set

$$\tilde{p}_v = \begin{cases} p_v \cup N_v, & v \text{ is a vertex of } K_0, \\ p_\sigma - \bigcup \{\text{Int } p_{v'}, v' \text{ is a vertex of } K_0\}, & v = \sigma, \\ p_\tau, & v = \tau. \end{cases}$$

Then $P_0 = \{\tilde{p}_v : v \text{ is a vertex of } K\}$ is a partition on S^m , and $T_0 : P_0 \rightarrow J_0$ defined by $T_0(\tilde{p}_v) = T(p_v)$ satisfies (A1)–(A3), (E1) and (E2). If $h_0 : S^m \rightarrow S^n$ is a map induced by $T_0 : P_0 \rightarrow J_0$, then h_0 and f are $\mathcal{U}$ -close, where $\mathcal{U} = \{\text{St}(j, J_0), j \in J_0\}$ is a closed cover of S^n such that each nonempty intersection of elements of $\mathcal{U}$ is an absolute retract (in fact, it is a PL ball). Hence (see [Wa 5]) h_0 and f are homotopic maps.

This finishes the proof of 3.1.

4. Monotone maps. In §3 we have shown that, in certain range of dimensions, there exist essential, monotone maps $f : S^m \rightarrow S^n$ that are one-to-one over the image of a dense set. In this section we show how to construct monotone (inessential) surjections $f : S^m \rightarrow S^n$ that are one-to-one over the image of a dense set for any $m, n \geq 3$. If $m > n \geq 4$, the existence of such maps follows from 3.2. Indeed, let $f_i : S^i \rightarrow S^{i-1}$ be a monotone surjection that is one-to-one over the image of a dense set ($i = n + 1, n + 2, \dots, m$). Let $g_i : S^i \rightarrow S^i$ be a homeomorphism intermingling the two pertinent (countable) dense subsets of S^i ($i = n + 1, n + 2, \dots, m - 1$). Then the composition $f_{n+1}g_{n+1} \cdots f_{m-1}g_{m-1}f_m : S^m \rightarrow S^n$ is a map with the desired properties. However, we want to present an independent proof that also works for $3 \leq m \leq n$ or $n = 3$.

THEOREM 4.1. *For any $m, n \geq 3$ there exists a monotone surjection $h : S^m \rightarrow S^n$ that is one-to-one over the image of a dense set.*

In the proof we need

LEMMA 4.2. *Let $h : X \rightarrow Y$ be a surjective map between compact metric spaces. If each nonempty open set in X contains a fiber of h , then h is one-to-one over the image of a dense set.*

PROOF. Suppose not. Let $F_\varepsilon = \bigcup \{h^{-1}(y) : y \in Y, \text{diam } h^{-1}(y) \geq \varepsilon\}$. Then F_ε is a closed set for any $\varepsilon > 0$, and $\bigcup \{F_\varepsilon, \varepsilon > 0\}$ has nonempty interior. By Baire's Category Theorem [Du, p. 250], there exists $\varepsilon > 0$ such that F_ε has nonempty interior. Let $U \subseteq F_\varepsilon$ be a nonempty open set with $\text{diam } U < \varepsilon$. Then U does not contain any fibers of h , contrary to the hypothesis.

PROOF OF 4.1. We construct a defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ with the following properties.

(M1) Each $p \in P_i$ is connected, $i = 0, 1, 2, \dots$

(M2) If $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P_i$ are mutually distinct elements, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} \cap p_{i(4)} = \emptyset$.

As in §3, we show first how to construct the triple $T_{i+1}: P_{i+1} \rightarrow J_{i+1}$ from the triple $T_i: P_i \rightarrow J_i$ already constructed. As before, we suppress indices, starting with an admissible function $T: P \rightarrow J$ satisfying (M1) and (M2). Let $\hat{T}: \hat{P} \rightarrow \tilde{J}$ be the triple constructed in §2. Observe that, although it is one-to-one, T will *never* be onto (this is the whole point of the construction).

We have to create new elements of P that will correspond to elements of $\tilde{J} - \text{Im } \hat{T}$, as well as make all elements of $\hat{P}$ connected. Observe that if $j \in \tilde{J}$ with $\text{Ind}(j) \leq 1$, then (by (A3)) $j \in \text{Im } \hat{T}$; and if $\text{Ind}(j) \geq 3$, then (by (M2)) $j \notin \text{Im } \hat{T}$.

First “connect up” all components of elements of $\hat{P}$. The only important property of $\hat{P}$ we use here is that for each disconnected $p_s \in \hat{P}$ there is a *connected* element $\hat{p} \in \hat{P}$ such that each component of p_s intersects $\hat{p}$. (If $p \in s$, $\hat{p} = p_{\{p\}}$ would do; see (H4).) Fixing a triangulation K' of S^m such that all pertinent subsets of S^m are (full) subcomplexes, for each disconnected $p \in \hat{P}$ choose a PL arc α_p lying in a connected element $c(p) \in \hat{P}$ such that $\alpha_p \cap \partial c(p)$ is a finite set intersecting each component of p . We can also arrange that different arcs are disjoint, and lie in the complement of the $(m-2)$ -skeleton of K' (here we use $m \geq 3$). Choose a subdivision K'' of K' such that α_p 's are subcomplexes with respect to K'' , and let N_p be the second derived neighborhood of α_p in $c(p)$. Finally, set

$$p^0 = \begin{cases} p \cup N_p & \text{if } p \in \hat{P} \text{ is disconnected,} \\ p - \bigcup \{N_{p'}: p' \in \hat{P} \text{ is disconnected}\} & \text{if } p \in \hat{P} \text{ is connected.} \end{cases}$$

Set $\hat{P}^0 = \{p^0, p \in \hat{P}\}$. Then $\hat{T}^0: \hat{P}^0 \rightarrow \tilde{J}$ defined by $\hat{T}^0(p_s^0) = \hat{T}(p_s)$ is a triple satisfying (A2), (A3), (M1) and (M2).

Now, we create new elements so that $\hat{T}^0$ can be extended to a bijection.

Let $\tilde{J} - \text{Im } \hat{T}^0 = \{j_1, j_2, \dots, j_r\}$. We can order this set so that $k \leq l$ implies $\text{Ind}(j_k) \leq \text{Ind}(j_l)$. We define a sequence $\{\hat{T}^k: \hat{P}^k \rightarrow \tilde{J}\}_{k=1}^r$ of triples satisfying (A2), (A3), (M1), (M2) and

(a_k) $\tilde{J} - \text{Im } \hat{T}^k = \{j_{k+1}, j_{k+2}, \dots, j_r\}$;

(b_k) for any $p \in P$, $p' \in \hat{P}^k$,

$$p \cap p' \neq \emptyset \Leftrightarrow \text{Int}(p \cap p') \neq \emptyset \Leftrightarrow T(p) \cap \hat{T}^k(p') \neq \emptyset.$$

Clearly, $\hat{T}^0: \hat{P}^0 \rightarrow \tilde{J}$ satisfies (a₀) and (b₀). Finally, we set $\{\tilde{T}: \tilde{P} \rightarrow \tilde{J}\} = \{\hat{T}^r: \hat{P}^r \rightarrow \tilde{J}\}$.

An argument for the inductive step is as follows. Assume the triple $\hat{T}^{k-1}: \hat{P}^{k-1} \rightarrow \tilde{J}$ satisfies (A2), (A3), (M1), (M2), (a_{k-1}) and (b_{k-1}). Define

$$B = \bigcup \{p \in \hat{P}^{k-1}: \hat{T}^{k-1}(p) \cap j_k \neq \emptyset\}.$$

Claim 1. B is connected.

Indeed, let $p_1, p_2 \in \hat{P}^{k-1}$ with $\hat{T}^{k-1}(p_i) \cap j_k \neq \emptyset$, $i = 1, 2$. Let v_1, v_2 be the two vertices of βL such that $\hat{T}^{k-1}(p_i) = \text{St}(v_i, \beta^2 L)$, $i = 1, 2$. Similarly, let v be

the vertex of βL , such that $j_k = \text{St}(v, \beta^2 L)$. Let $\sigma, \sigma_1, \sigma_2$ be the simplexes of L whose barycenters are v, v_1, v_2 respectively. Since $\text{St}(v, \beta^2 L) \cap \text{St}(v_i, \beta^2 L) \neq \emptyset$, σ and σ_i are comparable simplexes, $i = 1, 2$ (i.e. one is a face of the other). Choose a sequence $w_1, w_2, \dots, w_l$ of vertices of σ , such that w_1 is a vertex of σ_1 , and w_l is a vertex of σ_2 . Say, $\sigma_1 = \langle w_1, a_1, \dots, a_q \rangle$, $\sigma_2 = \langle w_l, b_1, \dots, b_u \rangle$. In the sequence $\langle w_1, a_1, \dots, a_{q-1}, a_q \rangle, \langle w_1, a_1, \dots, a_{q-1} \rangle, \dots, \langle w_1, a_1 \rangle, \langle w_1 \rangle, \langle w_1, w_2 \rangle, \langle w_2 \rangle, \langle w_2, w_3 \rangle, \langle w_3 \rangle, \dots, \langle w_{l-1}, w_l \rangle, \langle w_l \rangle, \langle w_l, b_1 \rangle, \dots, \langle w_l, b_1, \dots, b_u \rangle$ any two consecutive simplexes of L are comparable. Let $j^1, j^2, \dots, j^x$ be the corresponding sequence of elements of J (determined by the barycenters of the simplexes in the sequence). Note that (1) any two consecutive elements in this sequence intersect, (2) we can arrange the vertices of σ_1 and σ_2 so that each element in the sequence intersects j_k , and (3) by our choice of indexing elements of $J - \text{Im } \hat{T}^0$ and the fact that $\text{Ind}(j) \leq 1$ implies $j \in \text{Im } T^0$, each element in the sequence is in $\text{Im } \hat{T}^{k-1}$. Now by (A3) and (M1), $(\hat{T}^{k-1})^{-1}(j^1), (\hat{T}^{k-1})^{-1}(j^2), \dots, (\hat{T}^{k-1})^{-1}(j^x)$ is a sequence of connected sets whose union contains p_1, p_2 and itself is contained in B such that any two consecutive elements intersect. Hence, B is connected.

Claim 2. If $p \in P$ and if $T(p) \cap j_k \neq \emptyset$, then $\text{Int}(B \cap p) \neq \emptyset$.

Indeed, if $j_k = \text{St}(v, \beta^2 L)$, and if v is a barycenter of a simplex $\sigma = \langle a_1, \dots, a_l \rangle$ of L , then $T(p) = \text{St}(a_i, \beta L)$ for some i , $1 \leq i \leq l$. But then

$$p' = (\hat{T}^{k-1})^{-1}(\text{St}(a_i, \beta^2 L)) \subseteq B$$

and $\text{Int}(p' \cap p) \neq \emptyset$ (by (b_{k-1})).

Following the well-established pattern, once again choose a triangulation K_k of S^m such that all relevant subsets of S^m are (full) subcomplexes. Let α be a PL arc in $\text{Int } B$ that intersects all sets of the form p or $\text{Int}(B \cap p')$ for some $p \in \hat{P}^{k-1}$ with $\hat{T}^{k-1}(p) \cap j_k \neq \emptyset$ or some $p' \in P$ with $T(p') \cap j_k \neq \emptyset$. By Claims 1 and 2 such an arc exists. We can also arrange that it misses the $(m-2)$ -skeleton of K_k . Let K'_k be a subdivision of K_k such that α is a subcomplex, and let N be the second derived neighborhood of α (in B). Define $A(p) = p - \text{Int } N$ for all $p \in \hat{P}^{k-1}$. Setting $\hat{P}^k = \{A(p) : p \in \hat{P}^{k-1}\} \cup \{N\}$ and $\hat{T}^k(A(p)) = \hat{T}^{k-1}(p)$, $T^k(N) = j_k$ defines a triple $\hat{T}^k : \hat{P}^k \rightarrow \tilde{J}$. The reader should observe that this triple satisfies (A2), (A3), (M1), (M2), (a_k) and (b_k) .

We now proceed with the description of a number of improvements on the construction of a triple $\tilde{T} : \tilde{P} \rightarrow \tilde{J}$. For convenience, we use the following notation. If $p_{i(1)}, \dots, p_{i(t)} \in P$ with $\bigcap_{r=1}^t T(p_{i(r)}) \neq \emptyset$, then by $A(p_{i(1)}, \dots, p_{i(t)})$ we denote the element of $\tilde{P}$ such that

$$T(A(p_{i(1)}, \dots, p_{i(t)})) = \text{St}(v, \beta^2 L),$$

where v is the barycenter of the simplex of L whose vertices are determined by "centers" of $T(p_{i(1)}), \dots, T(p_{i(t)})$.

(i) Given $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P$ with $T(p_{i(1)}) \cap T(p_{i(2)}) \cap T(p_{i(3)}) \cap T(p_{i(4)}) \neq \emptyset$, we can arrange that there exists a PL arc $\alpha \subset \text{Int } p_{i(1)}$ such that α intersects the interior of each of the following elements of $\tilde{P}$, and no other elements of $\tilde{P}$: $A(p_{i(1)}), A(p_{i(2)}, p_{i(2)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$.

The trick is first to specify an arc $\alpha \subset \text{Int } p_{i(1)}$ that meets "right" elements of $\hat{P}$. In the process of "connecting up", we can choose arcs to miss α , and (choosing

a small triangulation of S^m) we can arrange that the “connecting tubes” miss α . Consequently, α hits the “right” elements of $\hat{P}^0$. In the inductive process of constructing partitions $\hat{P}^1, \dots, \hat{P}^r = \tilde{P}$, we can choose the relevant arcs either to hit or to miss α (according to the nature of the partition element that is about to be constructed).

(ii) Along with the hypotheses as in (i), assume that U is an open set in S^m and $U \cap p_{i(1)} \neq \emptyset$. Then we can arrange that α (which satisfies the conclusion of (i)) is contained in U .

Indeed, if α is any arc as in (i), we can find a PL homeomorphism $\psi: S^m \rightarrow S^m$ such that $\psi = \text{identity}$ off of $\text{Int } p_{i(1)}$ and $\psi(\alpha) \subset U$. Then $\hat{P}' = \{\psi(\tilde{p}): \tilde{p} \in \tilde{P}\}$ and $\alpha' = \psi(\alpha)$ satisfy all conclusions of (ii).

(iii) Given $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P$ as in (i) and an arc $\alpha \subset \bigcup_{t=1}^4 \text{Int } p_{i(t)}$ intersecting each $\text{Int } p_{i(t)}$, $t = 1, 2, 3, 4$, we can arrange that $A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$ is contained in a prechosen neighborhood U of α .

Using an argument of the same type as in (i), we can arrange that α has all properties as an arc serving as a guide for constructing $A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)})$. Then it remains to choose a small triangulation of S^m to get the required containment (in the inductive process, already “born” elements cannot “grow”).

The improvements (ii) and (iii), applied to

$$A(p_{i(1)}), A(p_{i(1)}, p_{i(2)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)}),$$

coupled together yield the following (here we use $n \geq 3$).

(iv) For any nonempty open set $U \subseteq S^m$, in the construction of the defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$, whenever P_i is given, we can arrange (by carefully choosing P_{i+1} and P_{i+2}) that P_{i+2} contains an element contained in U .

(v) Given $p \in P$ and an open set $U \subseteq S^m$ with $p \subseteq U$, we can arrange that $\text{St}(A(p), \tilde{P}) \subseteq U$.

Indeed, using (H5), we can arrange that $\text{St}(p, \hat{P}) \subseteq U$. In the “connecting up” process, we choose $c(p')$ to be $p_{\{p''\}}$ for some $p'' \in P$. In this way we get $\text{St}(\hat{p}, \hat{P}^0) \subseteq U$ where $\hat{p} = p_{\{p\}}$. Since $\hat{p}$ is connected, we have $\hat{p}^0 \subseteq \hat{p}$ and hence $\text{St}(\hat{p}^0, \hat{P}^0) \subseteq U$. Inductively assume that $\text{St}(\hat{p}^{k-1}, \hat{P}^{k-1}) \subseteq U$, where $\hat{p} \in \hat{P}^{k-1}$ “comes” from $\hat{p} \in \hat{P}$. If $j_k \cap \hat{T}^{k-1}(\hat{p}^{k-1}) = \emptyset$, we have $\text{St}(\hat{p}^k, \hat{P}^k) \subseteq \text{St}(\hat{p}^{k-1}, \hat{P}^{k-1}) \subseteq U$. So assume that $j_k \cap \hat{T}^{k-1}(\hat{p}^{k-1}) \neq \emptyset$. The corresponding set B defined along the inductive argument can be written as $B = B_1 \cup B_2$, where $B_1 = \bigcup\{p' \in \hat{P}^{k-1}: p' \subseteq B, p' \cap \hat{p}^{k-1} \neq \emptyset\}$, and $B_2 = \bigcup\{p' \in \hat{P}^{k-1}: p' \cap \hat{p}^{k-1} = \emptyset\}$. Then $B_1 \subseteq U$ and each $p' \subseteq B_2$ hits B_1 . Consequently, if we replace the set B in the inductive argument by the set $B' = B_1 \cup \{\text{the collection of all components of } B_2 \cap U \text{ that hit } B_1\}$, the constructed element N will be contained in U , and hence $\text{St}(\hat{p}^k, \hat{P}^k) \subseteq U$.

Improvements (iv) and (v) give the following:

Let $\{U_1, U_2, U_3, \dots\}$ be a countable basis of open sets for the topology on S^m . Then we can arrange that, for each i , there exists $p(i) \in P_{3i}$ with $\text{St}(p(i), P_{3i}) \subseteq U_i$.

Since by 2.1(i) each of the sets of the form $\text{St}(p, P)$, $p \in P$, contains a fiber of the map $h: S^m \rightarrow S^n$ determined by such a defining sequence, we conclude that h satisfies the hypotheses of 4.2, and hence it is one-to-one over the image of a dense set. (M1) implies that h is monotone.

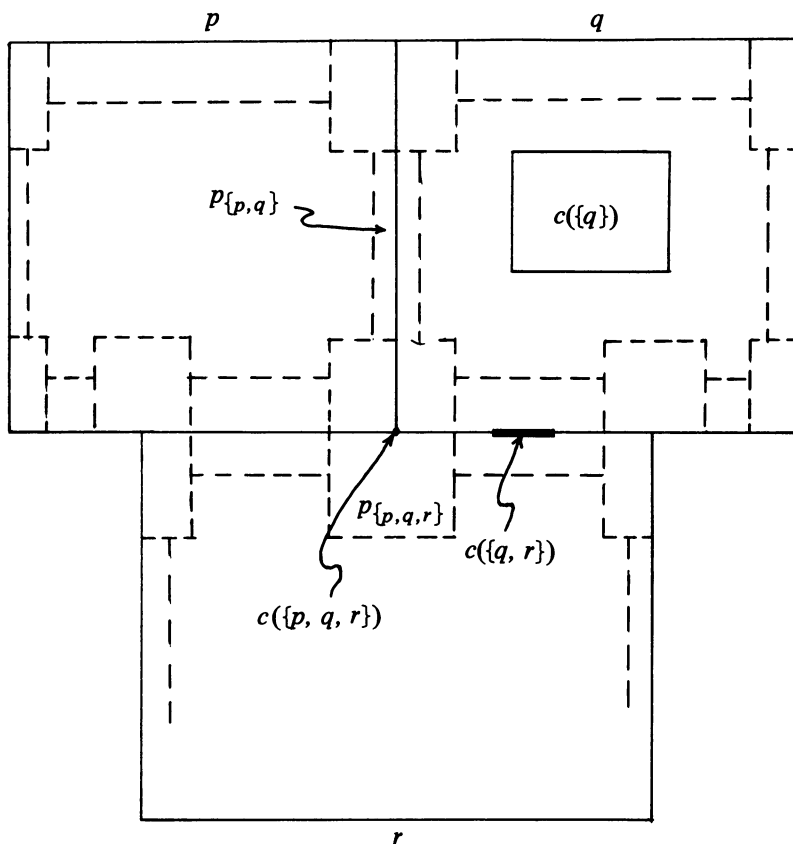


FIGURE 1

Let L' be any triangulation of S^{n-1} . Then there exists a partition P of S^{m-1} and an admissible map $T: P \rightarrow J$, $J = \{\text{St}(v, \beta L') : v \text{ is a vertex of } L'\}$ such that if $p_{i(1)}, p_{i(2)}, p_{i(3)} \in P$ are distinct, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} = \emptyset$. We now “suspend” the triple $T: P \rightarrow J$. Let J_0 be the standard handlebody decomposition of S^n corresponding to the triangulation $L = \Sigma L'$. P is a partition of $S^{m-1} \subset S^m$. Let P'_0 be the partition of S^m consisting of slightly “thickened” copies of $p \in P$ together with two m -balls corresponding to the suspension points. Defining $T'_0: P'_0 \rightarrow J_0$ in the obvious way, the reader should realize that this triple satisfies (A1)–(A3) and (M2). It remains to “connect up” elements of P'_0 . Observing that both m -balls in P'_0 , corresponding to two suspension points, intersect all components of all disconnected elements of P'_0 , the author leaves this as an exercise.

This completes the proof of 4.1. $\square$

REMARK 4.3. In [Be-Wa] it is shown that a map $f: S^m \rightarrow S^2$ that is one-to-one over the image of a dense set is far from being monotone. Using the technique of this section, one can construct a map $f: S^m \rightarrow S^2$ that is one-to-one over the image of a dense set, thus giving an alternative proof of the result in [Be-Wa]. One

finds a defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ with the additional property:

(U) If $p_{i(1)}, p_{i(2)}, p_{i(3)} \in P$ are mutually distinct, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} = \emptyset$.

The argument is slightly easier than the one given in the case of monotone maps, since one does not have to worry about “connecting up” various components.

REMARK 4.4. Maps $h: S^m \rightarrow S^n$ constructed in 4.1 and 4.3, are totally unstable (i.e. they have no stable values). In fact $h: S^m \rightarrow S^n$ can be approximated by a map $h_i: S^m \rightarrow S^n$ induced by the admissible function $T_i: P_i \rightarrow J_i$, which, in turn, can be approximated by a map $h'_i: S^m \rightarrow S^n$ induced by the triple $\hat{T}_i: \hat{P}_i \rightarrow \hat{J}_i$. If $T_i: P_i \rightarrow J_i$ satisfies (M2), then $\text{Im } h'_i$ is contained in the union of all elements j of J_i with $\text{Ind}(j) \leq 2$, and hence it is contained in a regular neighborhood of the 2-skeleton of $\beta^i L$. Consequently, $h: S^m \rightarrow S^n$ constructed as in 4.1 can be approximated by maps that factor through 2-dimensional polyhedra.

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